A REPRESENTABILITY THEOREM FOR SOME HUGE ABELIAN CATEGORIES

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ABSTRACT. We define quasi–locally presentable categories as big unions of coreflective subcategories which are locally presentable. Under appropriate hypotheses we prove a representability theorem for exact contravariant functors defined on a quasi–locally presentable category taking values in abelian groups. We show that the abelianization of a well generated triangulated category is quasi–locally presentable and we obtain a new proof of Brown representability theorem. Examples of functors which are not representable are also given.

INTRODUCTION

One of the main problems occurring in the theory of triangulated categories is to construct a left or right adjoint for a given triangulated functor. In his influential book on this subject, Neeman shows that the problem of finding an adjoint for a functor between triangulated categories may be equivalently studied at the level of abelianizations of these categories, where we have to construct an adjoint for some exact functor between abelian categories (see [11, Proposition 5.3.9]). Further Neeman considers in [11, Remark 5.3.10] that, unfortunately this idea is "nearly impossible" to be applied, since "existence theorems of adjoints usually depend on the categories being well–powered", that is one object must have only a set of subobjects (for an object of an abelian category this it equivalent to having only a set of quotients). But, in general, the abelianization of a triangulated category with arbitrary coproducts is huge, that is it does not satisfy the condition of being well (co)powered; see [11, Appendix C]. Hence the abelianization is often considered to be too big, hence not manageable (see also the Introduction of Krause's work [7]). This paper intends to change a little this perspective. More exactly, the result about the existence of adjoints depending on the categories being well powered is, obviously, the special Freyd's adjoint functor theorem: if $\mathcal{C}$ is a complete, well powered category having a cogenerator, then every functor $F : \mathcal{C} \to \mathcal{D}$ has a left adjoint if and only if it preserves limits. We argue that even if the abelianization of a well generated triangulated category is not always well (co)powered, it has enough structure allowing us to apply the general Freyd's adjoint functor

theorem: if $\mathcal{C}$ is a complete category, then every functor $F : \mathcal{C} \to \mathcal{D}$ has a left adjoint if and only if it preserves limits and satisfies the solution set condition (that is for every $D \in \mathcal{D}$ there is a set of objects $C_i \in \mathcal{C}$ and a set of maps $D \to F(C_i)$, $i \in I$ such that every map $D \to F(C)$, with $C \in \mathcal{C}$, factors through some $D \to F(C_i)$; see [1, 0.7]). The problem of the existence of the adjoints and the one of representability of a given functor are strongly related (to fix the settings, suppose that we work with preadditive categories): First, a functor $F : \mathcal{C} \to \mathcal{D}$ has a left adjoint if and only if the functor $\mathcal{D}(D, F(-)) : \mathcal{C} \to \mathcal{A}b$ is representable for all $D \in \mathcal{D}$; second a functor $F : \mathcal{C} \to \mathcal{A}b$ has a left adjoint if and only if it is representable (actually it is represented by the left adjoint evaluated at $\mathbb{Z}$).

The paper is organized as follows: In the first section, we recall the definition of the abelianization of a triangulated category and we show how the study of Brown representability may be done at the level of this abelianization. We show that the so called Freyd–style representability [12, Theorem 1.3] due to Neeman is a consequence of the original Freyd's adjoint functor theorem, but we also improve this result; whereas the old theorem requires the existence of a solution object for all cohomological functors which send coproducts into products, we ask this condition to be true only for the functor which we work with.

In the second section we define the notion of quasi–locally presentable category; it is a category which may be written as a union of some coreflective subcategories which are locally λ–presentable, where λ runs over all regular cardinals. Under appropriate hypotheses, we prove a representability theorem for exact, contravariant functors defined on such categories.

The third section is concerned with the abelianization of a well generated triangulated category. We show that this abelian category is quasi–locally presentable and satisfies the supplementary hypotheses allowing us to apply the representability theorem proved in the previous section. As a consequence we obtain a new proof of Brown representability theorem for well generated triangulated categories.

All categories which we work with are preadditive (enriched over $\mathcal{A}b$). Everywhere in our paper we may equally adopt the point of view of Gödel–Bernays–Von Neumann axiomatization of set theory, with its distinction between classes and sets, or to work in a given Grothendieck universe. In this last case, a set means a small set relative to that universe, whereas a class is a set which is not necessarily small.

1. Abelianization of a triangulated category and Brown representability

Consider a preadditive category $\mathcal{T}$. By a $\mathcal{T}$-*module* we understand a functor $X : \mathcal{T}^{\mathrm{op}} \to \mathcal{A}b$. Such a functor is called *finitely presentable* if there is an exact sequence of functors

$$\mathcal{T}(-, y) \to \mathcal{T}(-, x) \to X \to 0$$

for some $x, y \in \mathcal{T}$. Using Yoneda lemma, we know that the class of all natural transformations between two $\mathcal{T}$-modules X and Y denoted $\mathrm{Hom}_{\mathcal{T}}(X, Y)$ is actually a set, provided that X is finitely presentable. We consider the category $\mathrm{mod}(\mathcal{T})$ of all finitely presentable $\mathcal{T}$-modules, having $\mathrm{Hom}_{\mathcal{T}}(X, Y)$ as morphisms spaces, for all $X, Y \in \mathrm{mod}(\mathcal{T})$. The Yoneda functor

$$H = H_{\mathcal{T}} : \mathcal{T} \to \mathrm{mod}(\mathcal{T}) \text{ given by } H_{\mathcal{T}}(x) = \mathcal{T}(-, x)$$

is an embedding of $\mathcal{T}$ into $\mathrm{mod}(\mathcal{T})$, according to Yoneda lemma. If, in addition, $\mathcal{T}$ has coproducts then $\mathrm{mod}(\mathcal{T})$ is cocomplete and the Yoneda embedding preserves coproducts. It is also well–known (and easy to prove) that, if $F : \mathcal{T} \to \mathcal{A}$ is a functor into an additive category with cokernels, then there is a unique, up to a natural isomorphism, right exact functor $F^* : \mathrm{mod}(\mathcal{T}) \to \mathcal{A}$, such that $F = F^* H_{\mathcal{T}}$ (see [7, Lemma A.1]). Moreover, F preserves coproducts if and only if F^* preserves colimits.

In this section the category $\mathcal{T}$ will be triangulated with splitting idempotents. For definition and basic properties of triangulated categories the standard reference is [11]. Note that $\mathcal{T}$ has splitting idempotents, provided that $\mathcal{T}$ has countable coproducts, according to [11, Proposition 1.6.8]. Recall that $\mathcal{T}$ is supposed to be additive. A functor $\mathcal{T} \to \mathcal{A}$ into an abelian category $\mathcal{A}$ is called *homological* if it sends triangles into exact sequences. A contravariant functor $\mathcal{T} \to \mathcal{A}$ which is homological regarded as a functor $\mathcal{T}^{\mathrm{op}} \to \mathcal{A}$ is called *cohomological* (see [11, Definition 1.1.7 and Remark 1.1.9]). An example of a homological functor is the Yoneda embedding $H_{\mathcal{T}} : \mathcal{T} \to \mathrm{mod}(\mathcal{T})$. We know: $\mathrm{mod}(\mathcal{T})$ is an abelian category, and for every functor $F : \mathcal{T} \to \mathcal{A}$ into an abelian category, the unique right exact functor $F^* : \mathrm{mod}(\mathcal{T}) \to \mathcal{A}$ extending F is exact if and only if F is homological, by [5, Lemma 2.1]. This is the reason for which $\mathrm{mod}(\mathcal{T})$ is called the *abelianization* of the triangulated category $\mathcal{T}$ and is denoted sometimes by $\mathrm{A}(\mathcal{T})$. By [11, Corollary 5.1.23], $\mathrm{A}(\mathcal{T})$ is a Frobenius abelian category, with enough injectives and enough projectives, which are, up to isomorphism, exactly objects of the form $\mathcal{T}(-, x)$ for some $x \in \mathcal{T}$.

A cohomological functor $F : \mathcal{T} \to \mathcal{A}b$ may be also viewed as a homological functor $\mathcal{T} \to \mathcal{A}b^{\mathrm{op}}$, so it extends uniquely to a contravariant, exact functor $F^* : \mathrm{A}(\mathcal{T}) \to \mathcal{A}b$. It may be easily seen that $F^* \cong \mathrm{Hom}_{\mathcal{T}}(-, F)$. Thus we obtain:

Lemma 1.1. *If $\mathcal{T}$ is a triangulated category with splitting idempotents, then a cohomological functor $F : \mathcal{T} \to \mathcal{A}b$ is representable if and only if its extension $F^* : \mathrm{A}(\mathcal{T}) \to \mathcal{A}b$ is representable.*

Proof. If F is representable, then $F \in \mathrm{A}(\mathcal{T})$, so $F^* \cong \mathrm{Hom}_{\mathcal{T}}(-, F)$ is represented by F. Conversely if F^* is representable by an object in $\mathrm{A}(\mathcal{T})$ then this object must be isomorphic to F, therefore $F \in \mathrm{A}(\mathcal{T})$. Because F^* is exact, F must be injective, hence representable. $\square$

In order to give a more detailed picture related to the representability theorem at the level of abelianization, record the following:

Corollary 1.2. *Let $\mathcal{T}$ be a triangulated category with coproducts. The following are equivalent:*

    (i) *$\mathcal{T}$ satisfies Brown representability theorem.*

    (ii) *Every exact contravariant functor which sends coproducts into products $F : \mathrm{A}(\mathcal{T}) \to \mathcal{A}b$ is representable.*

    (iii) *Every covariant exact functor $F : \mathrm{A}(\mathcal{T}) \to \mathcal{A}$ which preserves colimits, with values into an abelian cocomplete category with enough injectives has a right adjoint.*

Proof. The equivalence (i)$\Leftrightarrow$(ii) follows by Lemma 1.1, whereas and the implication (iii)$\Rightarrow$(ii) is obvious, by replacing contravariant functors $\mathrm{A}(\mathcal{T}) \to \mathcal{A}b$ with covariant functors $\mathrm{A}(\mathcal{T}) \to \mathcal{A}b^{\mathrm{op}}$. Finally (i)$\Rightarrow$(iii) follows by [2, Theorem 1.1]. $\qquad\square$

Recall that the solution set condition for functors with values in the category of abelian groups $F : \mathcal{C} \to \mathcal{A}b$ may be stated as follows: there is a set of $\mathcal{S}$ of objects in $\mathcal{C}$ such that for any $C \in \mathcal{C}$ and any $y \in F(C)$ there are $S \in \mathcal{S}$, $y \in F(S)$ and $f : S \to C$ such that $F(x) = y$ (see [8, Chapter V, §6, Theorem 3]. By [13, Chapter 2, Theorem 1.1] this is further equivalent to the fact that there are objects $S_i \in \mathcal{C}$ indexed over a set I and a functorial epimorphism

$$\bigoplus_{i \in I} \mathcal{C}(S_i, -) \to F \to 0.$$

We say that F has a solution object provided that it has a solution set consisting of a single object. Obviously if $F \cong \mathcal{C}(C, -)$ is representable, then F has a solution object, namely $\mathcal{C}(C, -) \xrightarrow{\cong} F \to 0$.

Lemma 1.3. *If $\mathcal{T}$ is a triangulated category with splitting idempotents, then a cohomological functor $F : \mathcal{T} \to \mathcal{A}b$ has a solution object if and only if $F^* : \mathrm{A}(\mathcal{T}) \to \mathcal{A}b$ has a solution object.*

Proof. Suppose F has a solution object, i.e. there is a functorial epimorphism $H(t) = \mathcal{T}(-, t) \to F \to 0$, with $t \in \mathcal{T}$. In order to show that F^* has a solution object, we want to show that the induced natural transformation

$$\mathrm{Hom}_{\mathcal{T}}(-, H(t)) \to \mathrm{Hom}_{\mathcal{T}}(-, F) = F^*$$

is an epimorphism. That is, we want to show that the map

$$\mathrm{Hom}_{\mathcal{T}}(X, H(t)) \to \mathrm{Hom}_{\mathcal{T}}(X, F)$$

is surjective, for all $X \in \mathrm{A}(\mathcal{T})$. But every $X \in \mathrm{A}(\mathcal{T})$ admits an embedding $0 \to X \to H(x)$ into an injective object. Since $H(t) \in \mathrm{A}(\mathcal{T})$ is injective and F^* is exact, we obtain a diagram with exact rows:

$$
\begin{array}{ccccc}
\mathrm{Hom}_{\mathcal{T}}(H(x), H(t)) & \longrightarrow & \mathrm{Hom}_{\mathcal{T}}(X, H(t)) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
\mathrm{Hom}_{\mathcal{T}}(H(x), F) & \longrightarrow & \mathrm{Hom}_{\mathcal{T}}(X, F) & \longrightarrow & 0
\end{array} \quad .
$$

By Yoneda lemma we know that the first vertical map is isomorphic to $\mathcal{T}(x, t) \to F(x)$, hence it is surjective, thus the diagram above proves the direct implication.

Conversely if there is $X \in \mathrm{A}(\mathcal{T})$ and a natural epimorphism

$$\mathrm{Hom}_{\mathcal{T}}(-, X) \to \mathrm{Hom}_{\mathcal{T}}(-, F) \to 0,$$

then let $H(t) \to X \to 0$ be an epimorphism in $\mathrm{A}(\mathcal{T})$, with $t \in \mathcal{T}$. Consider the composed map

$$\mathrm{Hom}_{\mathcal{T}}(-, H(t)) \to \mathrm{Hom}_{\mathcal{T}}(-, X) \to \mathrm{Hom}_{\mathcal{T}}(-, F).$$

Evaluating it at $H(x)$ for an arbitrary $x \in \mathcal{T}$, we obtain a surjective map $\mathcal{T}(x, t) \to F(x)$, hence F has a solution object. $\square$

Theorem 1.4. *If $\mathcal{T}$ is a triangulated category with coproducts, then a cohomological functor, sending coproducts into products $F : \mathcal{T} \to \mathcal{A}b$ is representable if and only if it has a solution object.*

Proof. Under the hypotheses imposed on $\mathcal{T}$ and F, the abelian category $\mathrm{A}(\mathcal{T})$ is cocomplete and the induced functor $F^* : \mathrm{A}(\mathcal{T}) \to \mathcal{A}b$ sends colimits into limits. Therefore it is representable if and only if it has a solution object. Thus the theorem follows by combining Lemmas 1.1 and 1.3. $\square$

Remark 1.5. Theorem 1.4 says more than the Neeman's Freyd style representability theorem [12, Theorem 1.3]. Indeed the cited result states that if every cohomological functor which sends coproducts into products has a solution objects, then every such a functor is representable, whereas our result involves a fixed functor. Consequently versions of Brown representability theorem (see for example [9, Theorem 3.7]) which make use of this result may be improved.

2. Quasi–locally presentable abelian categories

We begin this section by recalling some definitions: A cardinal λ is said to be *regular* provided that it is infinite and it can not be written as a sum of less than λ cardinals, all smaller than λ. Denote by $\mathfrak{R}$ the class of all regular cardinals.

Let $\mathcal{A}$ be a additive category and $\mathcal{C} \subseteq \mathcal{A}$ be a subcategory. Let $F : \mathcal{A} \to \mathcal{A}b$ be a contravariant functor. The *category of elements* of $F|_{\mathcal{C}}$, where $F|_{\mathcal{C}}$ denotes the restriction of F at $\mathcal{C}$, is by definition constructed as follows:

$$\mathcal{C}/F = \{(X, x) \mid X \in \mathcal{C}, x \in F(C)\},$$

with the morphisms

$$\mathcal{C}/F((X_1, x_1), (X_2, x_2)) = \{\alpha \in \mathcal{C}(X_1, X_2) \mid F(\alpha)(x_2) = x_1\}.$$

In particular, for any object $A \in \mathcal{A}$, let denote

$$\mathcal{C}/A = \mathcal{C}/\mathcal{A}(-, A) = \{(C, \xi) \mid C \in \mathcal{C}, \xi : C \to A\},$$

$$\mathcal{C}/A((C_1, \xi_1), (C_2, \xi_2)) = \{\alpha \in \mathcal{C}(C_1, C_2) \mid \xi_2 \alpha = \xi_1\}.$$

Consider a regular cardinal λ. A non–empty category $\mathcal{S}$ is called λ–*filtered* if the following two conditions are satisfied:

F1. For every set $\{s_i \mid i \in I\}$ of less that λ objects of $\mathcal{S}$ there are an object $s \in \mathcal{S}$ and morphisms $s_i \to s$ in $\mathcal{S}$, for all $i \in I$.

F2. For every set $\{\sigma_i : s \to t \mid i \in I\}$ of less that λ morphisms in $\mathcal{S}$, there is a morphism $\tau : t \to u$ such that $\tau\sigma_i = \tau\sigma_j$, for all $i, j \in I$.

Let A be an object of a category $\mathcal{A}$. Then the functor $\mathcal{A}(A, -)$ preserves the colimit of a diagram $\mathcal{S} \to \mathcal{A}$, $s \mapsto X(s)$ in $\mathcal{A}$ (indexed over a category $\mathcal{S}$), if and only if every map $g : A \to \operatorname{colim}_{s \in \mathcal{S}} X(s)$ factors as

$$
\begin{array}{ccc}
A & & \\
\downarrow{\scriptstyle f} & \searrow{\scriptstyle g} & \\
X(u) & \xrightarrow[\xi_u]{} & \operatorname{colim}_{s \in \mathcal{S}} X(s)
\end{array}
$$

through some canonical map ξ_u with $u \in \mathcal{S}$, and every such factorization is essentially unique, in the sense that if $f_1, f_2 : A \to X(u)$ with $\xi_u f_1 = g = \xi_u f_2$ then there is $\sigma : u \to t$ a map in $\mathcal{S}$ such that $X(\sigma)f_1 = X(\sigma)f_2$. The object $A \in \mathcal{A}$ is called λ–*presentable* if $\mathcal{A}(A, -)$ preserves all λ–filtered colimits. The category $\mathcal{A}$ is called *locally λ–presentable* provided that it is cocomplete, and has a set $\mathcal{S}$ of λ–presentable objects such that every $X \in \mathcal{A}$ is a λ–filtered colimit of objects in $\mathcal{S}$ (see [1, Definition 1.17], but also [1, Remark 1.21])). Note that, if $\mathcal{A}$ is locally λ–presentable, then the subcategory $\mathcal{A}^\lambda$ of all λ–presentable objects in $\mathcal{A}$ is essentially small, and for every object $A \in \mathcal{A}$, the category $\mathcal{A}^\lambda/A$ is λ–filtered and

$$
A \cong \operatorname*{colim}_{(X,\xi) \in \mathcal{A}^\lambda/A} X,
$$

as we may see from [1, Proposition 1.22]. A category is called *locally presentable* if it is locally λ–presentable for some regular cardinal λ.

Remark 2.1. Let $\mathcal{A}$ be a locally λ–presentable category, and let $F : \mathcal{A} \to \mathcal{A}b$ be a contravariant functor which sends colimits into limits. By Freyd's special adjoint functor theorem, the functor F is representable, so $F \cong \mathcal{A}(-, A)$, for some $A \in \mathcal{A}$. Thus the categories $\mathcal{A}^\lambda/A$ and $\mathcal{A}^\lambda/F$ are isomorphic, so

$$
F \cong \mathcal{A}(-, \operatorname*{colim}_{(X,x) \in \mathcal{A}^\lambda/F} X).
$$

Recall that we call *cofinal* a subcategory $\mathcal{S}$ of a category $\mathcal{C}$ satisfying the following two properties: For every $c \in \mathcal{C}$ there is a map $c \to s$ in $\mathcal{C}$ for some $s \in \mathcal{S}$; and for any two maps $c \to s_1$ and $c \to s_2$ in $\mathcal{C}$, with $s_1, s_2 \in \mathcal{S}$ there are $s \in \mathcal{S}$ and two maps $s_1 \to s$ and $s_2 \to s$ in $\mathcal{S}$ such that the composed morphisms $c \to s_1 \to s$ and $c \to s_2 \to s$ are equal. It is well–known that if $\mathcal{S}$ is a cofinal subcategory of $\mathcal{C}$, then colimits over $\mathcal{C}$ and colimits over $\mathcal{S}$ coincide (see [1, 0.11]).

Lemma 2.2. *Let $\mathcal{A}$ be an abelian category, and let $F : \mathcal{A} \to \mathcal{A}b$ be a contravariant, exact functor. Let $\mathcal{C} \subseteq \mathcal{A}$ be a subcategory closed under finite*

coproducts and cokernels. If $\mathcal{S}$ is a subcategory of $\mathcal{C}$ closed under finite coproducts and satisfying the property that every $X \in \mathcal{C}$ admits an embedding $0 \to X \to S$ into an object in $\mathcal{S}$, then $\mathcal{S}/F$ is a cofinal subcategory of $\mathcal{C}/F$.

Proof. Let $(X, x) \in \mathcal{C}/F$. Consider an embedding $0 \to X \overset{\alpha}{\to} S$, with $S \in \mathcal{S}$. Thus $F(S) \overset{F(\alpha)}{\longrightarrow} F(X) \to 0$ is exact, showing that there exists $y \in F(S)$ with $F(\alpha)(y) = x$. Therefore α is a map in $\mathcal{C}/F$ between (X, x) and (S, y).

Now we claim that if $\alpha : X_1 \to X_2$ is a map in $\mathcal{C}$, and $x_2 \in f(X_2)$ is an element with the property $F(\alpha)(x_2) = 0$, then there is a morphism $\gamma \in \mathcal{C}/F((X_2, x_2), (S, y))$ into an object $(S, y) \in \mathcal{S}/F$ such that $\gamma\alpha = 0$. Indeed consider X being defined by exact sequence $X_1 \overset{\alpha}{\to} X_2 \overset{\beta}{\to} X \to 0$. Since the sequence of abelian groups $0 \to F(X) \overset{F(\beta)}{\to} F(X_2) \overset{F(\alpha)}{\to} F(X_3)$ is also exact and $F(\alpha)(x_2) = 0$, we obtain an element $x \in F(X)$ such that $F(\beta)(x) = x_2$. For obtaining the required γ, compose β with a morphism in $\mathcal{C}/F$ from (X, x) into an object (S, y), which is constructed as in the first part of this proof.

Finally for two morphisms

$$\alpha_1 \in \mathcal{C}/F((X, x), (S_1, y_1)) \text{ and } \alpha_2 \in \mathcal{C}/F((X, x), (S_2, y_2)),$$

denote by ρ_1 and ρ_2 the respective injections of the coproduct $S_1 \amalg S_2$. Then $F(\rho_1\alpha_1 - \rho_2\alpha_2)(y_1, y_2) = x - x = 0$, so our claim for $\alpha = \rho_1\alpha_1 - \rho_2\alpha_2$ gives a morphism $(S_1 \amalg S_2, (y_1, y_2)) \to (S, y)$ in $\mathcal{C}/F$, with $S \in \mathcal{S}$, such that the composed morphisms $X \to S_1 \to S_1 \amalg S_2 \to S$ and $X \to S_2 \to S_2 \amalg S_2 \to S$ are equal. $\square$

We call *quasi–locally presentable* a category $\mathcal{A}$ which is a union

$$\mathcal{A} = \bigcup_{\lambda \in \mathfrak{R}} \mathcal{A}_\lambda,$$

such that the subcategory $\mathcal{A}_\lambda$ locally λ–presentable and closed under colimits in $\mathcal{A}$, for any $\lambda \in \mathfrak{R}$. Denote by $I_\lambda : \mathcal{A}_\lambda \to \mathcal{A}$ the inclusion functor, which preserves colimits by our assumption. Note that by Freyd's special adjoint functor theorem, the subcategory $\mathcal{A}_\lambda$ is coreflective, that is I_λ has a right adjoint $R_\lambda : \mathcal{A} \to \mathcal{A}_\lambda$. For a quasi–locally presentable category $\mathcal{A}$ and a regular cardinal λ we denote by $\mathcal{A}_\lambda^\lambda$ the subcategory of all λ–presentable objects of $\mathcal{A}_\lambda$. Note that if $\mathcal{A}$ is locally presentable, more precisely it is locally κ–presentable for some regular cardinal κ, then it is also quasi–locally presentable, with $\mathcal{A}_\lambda = \mathcal{A}$, for all regular cardinals $\lambda \geq \kappa$. In the sequel we shall work with quasi–locally presentable categories which are abelian. Note that in this case the coreflective subcategory $\mathcal{A}_\lambda$ will be also abelian. Fix a regular cardinal κ. We say that a quasi–locally presentable abelian category $\mathcal{A}$ is weakly κ-generated if $\mathcal{A}$ coincide with its smallest full subcategory containing $\mathcal{A}_\kappa$ and being closed under kernels, cokernels, extensions and countable coproducts. Finally denote

$$\text{Inj}_\lambda \mathcal{A} = \{S \in \mathcal{A} \mid S \text{ is injective and } S \in \mathcal{A}_\lambda^\lambda\}.$$

Theorem 2.3. *Let $\mathcal{A}$ be a quasi–locally presentable, abelian category which is weakly κ–generated, for some regular cardinal κ. Suppose also that, for any regular cardinal $\lambda \geq \kappa$, every $X \in \mathcal{A}_\lambda^\lambda$ admits an embedding $0 \to X \to S$ into an object $S \in \mathrm{Inj}_\lambda \mathcal{A}$. Then every exact, contravariant functor $F : \mathcal{A} \to \mathcal{A}b$ which sends coproducts into products is representable (necessarily by an injective object).*

Proof. For any $\lambda \in \mathfrak{R}$, consider the corresponding coreflective locally λ–presentable subcategory $I_\lambda : \mathcal{A}_\lambda \leftrightarrows \mathcal{A} : R_\lambda$.

Denote by $\mathcal{C}_0$ a skeleton of $\mathcal{A}_\kappa^\kappa$, and let $C_0 = \coprod_{(U,u)\in\mathcal{C}_0/F} U$. Let λ be a regular cardinal such that

$$\lambda \geq \kappa + \mathrm{card}\,\mathcal{C}_0 + \sum_{U\in\mathcal{C}_0} \mathrm{card}\,F(U) + \sum_{U\in\mathcal{C}_0} \mathrm{card}\,\mathcal{A}(U,C_0) + \aleph_1,$$

and denote by $\mathcal{C}$ a skeleton of $\mathcal{A}_\lambda^\lambda$.

Fix an exact contravariant functor which sends coproducts into products $F : \mathcal{A} \to \mathcal{A}b$. Then $FI_\lambda : \mathcal{A}_\lambda \to \mathcal{A}b$ sends colimits into limits, so $FI_\lambda \cong \mathcal{A}_\lambda(-, F_\lambda)$ for some $F_\lambda \in \mathcal{A}_\lambda$, for which follows by Remark 2.1 that

$$F_\lambda = \operatorname*{colim}_{(X,x)\in\mathcal{C}/F} X = \operatorname*{colim}_{(X,\xi)\in\mathcal{C}/F_\lambda} X,$$

with the canonical maps $\gamma_{(X,x)} : X \to F_\lambda$. We claim that

$$F(A) \cong \operatorname*{colim}_{(X,x)\in\mathcal{C}/F} \mathcal{A}(A,X),$$

for all $A \in \mathcal{A}_\kappa$. Since $A = I_\kappa(A) = I_\lambda(A)$ this means precisely that A preservers the colimit of the diagram $\mathcal{C}/F \to \mathcal{A}$, $(X,x) \mapsto X$. In order to prove our claim, consider in the first step that A is a coproduct of objects in $\mathcal{C}_0$, that is $A = \coprod_{i\in I} U_i$, for some set I, and some $U_i \in \mathcal{C}_0$. Denote by $j_i : U_i \to A$, $(i \in I)$ the canonical injections. Let $g : A \to F_\lambda$ be a map in $\mathcal{A}$. Since for all $U \in \mathcal{C}_0$ we have $U \in \mathcal{A}_\kappa \subseteq \mathcal{A}_\lambda$, we may identify $\mathcal{C}_0/F$ with $\mathcal{C}_0/F_\lambda$ thus $C_0 = \coprod_{(U,v)\in\mathcal{C}_0/F_\lambda} U$ with the canonical injections $\epsilon_{(U,v)} : U \to C_0$. Since $gj_i \in \mathcal{A}(U_i, F_\lambda)$ we get a unique $f : A \to C_0$, such that $fj_i = \epsilon_{(U_i, gj_i)}$ from the universal property of the coproduct. Observe that for $c_0 = (v)_{(U,v)\in\mathcal{C}_0/F_\lambda}$ our choice of $\lambda \geq \sum_{U\in\mathcal{C}_0} \mathrm{card}\,F(U)$ assures us that $(C_0, c_0) \in \mathcal{A}_\lambda^\lambda/F_\lambda$. Thus we may assume that $(C_0, c_0) \in \mathcal{C}/F_\lambda$. Moreover by construction $\gamma_{(C_0,c_0)}f = g$, so g factors through $\gamma_{(C_0,c_0)}$.

It remains to show that this factorization is essentially unique. Consider therefore two maps $f_1, f_2 : A \to C_0$ such that $\gamma_{(C_0,c_0)}f_1 = g = \gamma_{(C_0,c_0)}f_2$. Denote $\mathcal{N} = \{(U,h) \mid U \in \mathcal{C}, h \in \mathcal{A}(U,C_0)$ with $\gamma_{(C_0,c_0)}h = 0\}$, and put $C_1 = \coprod_{(U,h)\in\mathcal{N}} U$ with the canonical injections $k_{(U,h)} : U \to C_1$. By the choice of λ we have $\lambda \geq \mathrm{card}\,\mathcal{A}(U,C_0) \geq \mathrm{card}\,\mathcal{N}$, hence $(C_1, 0) \in \mathcal{A}_\lambda^\lambda/F$. Again we may consider $(C_1, 0) \in \mathcal{C}/F$. We have $(U_i, (f_1-f_2)j_i) \in \mathcal{N}$, hence there is a unique $\theta : A \to C_1$ such that $\theta j_i = k_{(U_i, (f_1-f_2)j_i)}$ for all $i \in I$. Further there is a unique morphism $\eta : C_1 \to C_0$ such that $\eta k_{(U,h)} = \epsilon_{(U,0)}$ for all $(U,h) \in \mathcal{N}$. Clearly η is a map in $\mathcal{C}/F$ between $(C_1, 0)$ and (C_0, c_0).

If $C_1 \xrightarrow{\eta} C_0 \xrightarrow{\delta} C \to 0$ is exact in $\mathcal{A}$ we deduce that $C \in \mathcal{C}$, because $\mathcal{C}$ is closed under cokernels (see [1, Proposition 1.16]), and from the exactness of F we infer that there is $c \in F(C)$ such that $F(\delta)(c) = c_0$. Thus $\delta : (C_0, c_0) \to (C, c)$ in $\mathcal{C}/F$, and $\delta(f_1 - f_2) = \delta\eta\theta = 0$, finishing the proof of the first step of our claim.

Now for an arbitrary $A \in \mathcal{A}_\lambda$, note that A is a colimit of objects in $\mathcal{C}_0$, so it is a cokernel of the form $A_1 \to A_0 \to A \to 0$ with A_1 and A_0 being coproducts of objects in $\mathcal{C}_0$. Using the first step before, we get easily $\mathcal{A}(A, F_\lambda) \cong \mathrm{colim}_{(X,x)\in\mathcal{C}/F} \mathcal{A}(A, X)$ canonically.

Finally, there is an obvious natural transformation

$$\phi : \mathop{\mathrm{colim}}_{(X,x)\in\mathcal{A}_\lambda^\lambda/F} \mathcal{A}(-, X) \to F.$$

We know that $\mathcal{A}_\lambda^\lambda/F$ is λ–filtered (see [4, Korollar 5.4]), hence colimits of abelian groups indexed over this category are exact and commutes with products of less that λ objects (see [4, Satz 5.2]). Since every $X \in \mathcal{A}_\lambda^\lambda$ admits an embedding in an object $S \in \mathrm{Inj}_\lambda \mathcal{A}$, Lemma 2.2 tells us that $\mathrm{Inj}_\lambda \mathcal{A}/F$ is a cofinal subcategory of $\mathcal{A}_\lambda^\lambda/F$, so $\mathrm{colim}_{(X,x)\in\mathcal{A}_\lambda^\lambda/F} \mathcal{A}(-, X) \cong \mathrm{colim}_{(S,s)\in\mathrm{Inj}_\lambda \mathcal{A}/F} \mathcal{A}(-, S)$ is an exact functor. Moreover, for all $A \in \mathcal{A}_\kappa$ we have

$$\mathop{\mathrm{colim}}_{(X,x)\in\mathcal{A}_\lambda^\lambda/F} \mathcal{A}(A, X) \cong \mathop{\mathrm{colim}}_{(X,x)\in\mathcal{C}/F} \mathcal{A}(A, X) \cong F(A)$$

canonically, as we have seen before. We infer that the full subcategory $\{A \in \mathcal{A} \mid \phi_A \text{ is an isomorphism}\} \subseteq \mathcal{A}$ is closed under kernels, cokernels, extensions, countable coproducts (since $\lambda \geq \aleph_1$) and contains $\mathcal{A}_\kappa$, hence it is equal to $\mathcal{A}$ using the hypothesis of weak κ–generation. This means that $\mathcal{C}$ forms a solution set for F, so F is representable by the general Freyd's adjoint functor theorem. $\qquad\square$

Remark 2.4. With the notations made in Theorem 2.3 and its proof, the argument using to show the fact that $\mathcal{A}(A, F_\lambda) \cong \mathrm{colim}_{(X,x)\in\mathcal{C}/F} \mathcal{A}(A, X)$, for $A = \coprod_{i\in I} U_i$, with $U_i \in \mathcal{C}_0$ is inspired by [3, Lemma 2.11]. However, we did't only change the settings, but we also improved the proof of Franke. A simple translation of his argument in our settings would require the condition $\mathrm{card}\,\mathcal{A}(U, X) \leq \lambda$ for all $U \in \mathcal{A}_\kappa^\kappa$ and all $X \in \mathcal{A}_\lambda^\lambda$. A priori is not clear how we may choose such a regular cardinal λ. Instead this, we required $\sum_{U\in\mathcal{C}_0} \mathrm{card}\,\mathcal{A}(U, C_0) \leq \lambda$, where the left hand side of this inequality does't depend of λ.

Example 2.5. In Theorem 2.3 the exactness of the functor $F : \mathcal{A} \to \mathcal{A}b$ (which sends coproducts into products) is an essential hypothesis. More precisely, the weaker requirement that F sends colimits into limits is not sufficient to conclude that it is representable. Using an idea from [10] we may to construct a non–representable functor $F : \mathcal{A} \to \mathcal{A}b$, which sends colimits into limits. For this purpose suppose that the quasi–locally presentable category $\mathcal{A}$ from the Theorem 2.3 is not locally presentable, that is $\mathcal{A} \neq \mathcal{A}_\lambda$

for every $\lambda \in \mathfrak{R}$. The fact that $\mathcal{A}$ is weakly generated which is used in combination with the exactness of F doesn't play any role in this example. Since R_λ is not an equivalence, we may find an object $0 \neq X_\lambda \in \mathcal{A}$ such that $R_\lambda(X_\lambda) = 0$, for every $\lambda \in \mathfrak{R}$. Strictly speaking we need here a version of axiom of choice which works for proper classes. The functor

$$F = \prod_{\lambda \in \mathfrak{R}} \mathcal{A}(-, X_\lambda)$$

is well defined since for every $X \in \mathcal{A}$ we have $X \in \mathcal{A}_\kappa$ for some $\kappa \in \mathfrak{R}$, so $\mathcal{A}(X, X_\lambda) = 0$ for all $\lambda \geq \kappa$. It is easy to see that this functor does the job we claim.

3. The abelianization of a well generated triangulated category

The main purpose of this section is to show that the abelianization of a triangulated category which is well–generated in the sense of Neeman is quasi–locally presentable and satisfies the hypothesis of Theorem 2.3. Consequently we obtain a new proof of Brown representability theorem for such triangulated categories.

Consider a regular cardinal λ. We call λ–*(co)product* a (co)products of less that λ objects. An category $\mathcal{C}$ is called λ–*cocomplete* if $\mathcal{C}$ has λ–coproducts and cokernels. It is easy to see that $\mathcal{C}$ is λ–cocomplete if and only if it contains all colimits of diagrams with less that λ morphisms. A $\mathcal{C}$-module over a λ–cocomplete category is called λ–*left exact* if it is left exact and sends λ–coproducts into products. Provided that the category $\mathcal{C}$ is essentially small, the class $\mathrm{Hom}_\mathcal{C}(X, Y)$ is actually a set for all $X, Y \in \mathcal{C}$. Thus we are allowed to consider the category $\mathrm{Mod}(\mathcal{C})$ of all $\mathcal{C}$-modules. If $\mathcal{C}$ is also λ–cocomplete, then denote by $\mathrm{Lex}_\lambda(\mathcal{C}^{\mathrm{op}}, \mathcal{A}b)$ the full subcategory of $\mathrm{Mod}(\mathcal{C})$ consisting of λ–left exact modules. We know that $\mathrm{Lex}_\lambda(\mathcal{C}^{\mathrm{op}}, \mathcal{A}b)$ is a locally λ–presentable category, and the embedding $\mathcal{C} \to \mathrm{Lex}_\lambda(\mathcal{C}^{\mathrm{op}}, \mathcal{A}b)$ given by $X \mapsto \mathcal{C}(-, X)$ identifies $\mathcal{C}$, up to isomorphism, with the subcategory of λ–presentable objects in $\mathrm{Lex}_\lambda(\mathcal{C}^{\mathrm{op}}, \mathcal{A}b)$ (see [4, Korollar 7.9]).

As before, let λ denote a regular cardinal. If $\mathcal{S}$ is an preadditive, essentially small category with λ–coproducts, denote by $\mathrm{Prod}_\lambda(\mathcal{S}^{\mathrm{op}}, \mathcal{A}b)$ the full subcategory of $\mathrm{Mod}(\mathcal{S})$, consisting of those modules which preserves λ–products. Clearly a finitely presentable $\mathcal{S}$-module, that is an element in $\mathrm{mod}(\mathcal{S})$, preserves arbitrary products, hence it belongs to $\mathrm{Prod}_\lambda(\mathcal{S}^{\mathrm{op}}, \mathcal{A}b)$.

Lemma 3.1. *For a regular cardinal λ, consider an additive, essentially small category $\mathcal{S}$ having λ–coproducts. Then $\mathrm{Prod}_\lambda(\mathcal{S}^{\mathrm{op}}, \mathcal{A}b)$ is a locally λ–presentable category, and the embedding $\mathrm{mod}(\mathcal{S}) \overset{\subseteq}{\longrightarrow} \mathrm{Prod}_\lambda(\mathcal{S}^{\mathrm{op}}, \mathcal{A}b)$ identifies $\mathrm{mod}(\mathcal{S})$ with the full subcategory of $\mathrm{Prod}_\lambda(\mathcal{S}^{\mathrm{op}}, \mathcal{A}b)$ consisting of all λ–presentable objects.*

Proof. The category $\mathrm{mod}(\mathcal{S})$ has obviously λ–coproducts and cokernels, so it is λ–cocomplete. According to [7, Lemma B.1], there is an equivalence of

categories

$$\mathrm{Lex}_\lambda(\mathrm{mod}(\mathcal{S})^{\mathrm{op}}, \mathcal{A}b) \to \mathrm{Prod}_\lambda(\mathcal{S}^{\mathrm{op}}, \mathcal{A}b), \ X \mapsto XH_{\mathcal{S}},$$

where $H_{\mathcal{S}} : \mathcal{S} \to \mathrm{mod}(\mathcal{S})$ denotes the Yoneda functor. Thus $\mathrm{Prod}_\lambda(\mathcal{S}^{\mathrm{op}}, \mathcal{A}b)$ is locally λ–presentable. Further, the identification of λ–presentable objects in $\mathrm{Prod}_\lambda(\mathcal{S}^{\mathrm{op}}, \mathcal{A}b)$ follows by discussion above concerning λ–presentable objects in $\mathrm{Lex}_\lambda(\mathcal{C}^{\mathrm{op}}, \mathcal{A}b)$. $\square$

Let $\mathcal{T}$ be a triangulated category with coproducts. We need the following definitions: For regular cardinal λ, a λ–*localizing* subcategory of $\mathcal{T}$ ia a triangulated subcategory closed under λ–coproducts. A *localizing* subcategory is a subcategory which is λ–localizing, for all λ. We say that $\mathcal{T}$ is generated (in the triangulated sense) by a set of objects $\mathcal{S} \subseteq \mathcal{T}$, provided that an object $t \in \mathcal{T}$ vanishes, whenever $\mathcal{T}(s, t) = 0$ for all $s \in \mathcal{S}$. Further we say that $\mathcal{T}$ is perfectly generated by the set of objects $\mathcal{S}$ if $\mathcal{S}$ generates $\mathcal{T}$ and, for any $s \in \mathcal{S}$, the map $\mathcal{T}(s, \coprod_{i \in I} x_i) \to \mathcal{T}(s, \coprod_{i \in I} y_i)$ is surjective, for every set of maps $\{x_i \to y_i \mid i \in I\}$ such that $\mathcal{T}(s, x_i) \to \mathcal{T}(s, y_i)$ is surjective, for all $i \in I$. Finally $\mathcal{T}$ is called *well λ–generated*, where $\lambda \in \mathfrak{R}$, provided that $\mathcal{T}$ is perfectly generated by a set of objects which are also λ–*small*, that is, every map $s \to \coprod_{i \in I} x_i$, with $s \in \mathcal{S}$, factors trough a coproduct $\coprod_{i \in I'} x_i$ with card $I' \le \lambda$; the category $\mathcal{T}$ is well generated if it is well λ–generated, for some λ. Following [6, Theorem A], this definition is equivalent to the original one given by Neeman. Note that, by [9, Corollary 2.6], if $\mathcal{T}$ is perfectly generated by $\mathcal{S}$, then $\mathcal{T}$ coincide with its smallest $\aleph_1$–localizing subcategory which contains arbitrary coproducts of objects in $\mathcal{S}$.

Suppose that $\mathcal{T}$ is well κ–generated triangulated category, having a perfectly generating set $\mathcal{S}$ consisting from κ-small objects. For any $\lambda \ge \kappa$ we consider the smallest λ localizing subcategory of $\mathcal{T}$ which contains $\mathcal{S}$ and denote it by $\mathcal{T}^\lambda$. Call λ–*compact* objects in $\mathcal{T}^\lambda$. By [6, Lemma 5] the category of λ–compact objects in $\mathcal{T}$ is independent of $\mathcal{S}$. Clearly it is essentially small and a skeleton of $\mathcal{T}^\lambda$ generates $\mathcal{T}$. Denote $\mathrm{A}_\lambda(\mathcal{T}) = \mathrm{Prod}_\lambda((\mathcal{T}^\lambda)^{\mathrm{op}}, \mathcal{A}b)$, for $\lambda \ge \kappa$ and $\mathrm{A}_\lambda(\mathcal{T}) = 0$ otherwise. We know by [11, Proposition A.1.8] that $\mathrm{A}_\lambda(\mathcal{T})$ is locally λ–presentable, and by [11, Proposition 6.5.3] that it is a coreflective subcategory of $\mathrm{A}(\mathcal{T})$, the restriction functor $R_\lambda : \mathrm{A}(\mathcal{T}) \to \mathrm{A}_\lambda(\mathcal{T})$ providing a right adjoint of the inclusion $I_\lambda : \mathrm{A}_\lambda(\mathcal{T}) \to \mathrm{A}(\mathcal{T})$.

Proposition 3.2. *Fix a regular cardinal $\kappa > \aleph_0$. If $\mathcal{T}$ is a well κ–generated triangulated category, then $\mathrm{A}(\mathcal{T})$ is a quasi–locally presentable abelian category which is weakly κ–generated.*

Proof. Denote by $\mathcal{A}$ the smallest subcategory of $\mathrm{A}(\mathcal{T})$ which is closed under kernels, cokernels, extensions, countable coproducts and contains $\mathrm{A}_\kappa(\mathcal{T})$. Let us show that $\mathrm{A}(\mathcal{T}) = \mathcal{A}$. Observe first that if $T \to U \to X \to Y \to Z$

is an exact sequence with $T, U, Y, Z \in \mathcal{A}$ then we can construct the commutative diagram with exact rows and column

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
T & \longrightarrow & U & \longrightarrow & X' & \longrightarrow & 0 & & \\
\| & & \| & & \downarrow & & & & \\
T & \longrightarrow & U & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z \\
 & & & & \downarrow & & \| & & \| \\
0 & \longrightarrow & X'' & \longrightarrow & Y & \longrightarrow & Z & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}
$$

showing that $X \in \mathcal{A}$. Therefore if $x \to y \to z \rightsquigarrow$ is a triangle in $\mathcal{T}$ with $H(x), H(z) \in \mathcal{A}$ then $H(y) \in \mathcal{A}$. It is shown in [9, Theorem 2.5] that every object $x \in \mathcal{T}$ is isomorphic to a homotopy colimit of a tower $x^0 \to x^1 \to \cdots$ such that $x^0 = 0$ and for every $n \in \mathbb{N}$ we have a triangle $p_n \to x^n \to x^{n+1} \rightsquigarrow$ with p_n being a coproduct of objects in $\mathcal{T}^\kappa$. Inductively $H(x^n) \in \mathcal{A}$, for all $n \in \mathbb{N}$, hence $H(\coprod_{n \in \mathbb{N}} x^n) \cong \coprod_{n \in \mathbb{N}} H(x^n) \in \mathcal{A}$, and finally $H(x) \in \mathcal{A}$. Now, for every $X \in \mathrm{A}(\mathcal{T})$ there is an exact sequence $H(y) \to H(x) \to X \to 0$, with $x, y \in \mathcal{T}$, thus $X \in \mathcal{A}$.

Note that we have already shown that $\mathcal{T}$ coincide with its smallest localizing subcategory which contains a skeleton of $\mathcal{T}^\kappa$. Therefore the proof of [11, Proposition 8.4.2] (more precisely [11, 8.4.2.3]) works for our case, hence $\mathcal{T} = \bigcup_{\lambda \geq \kappa} \mathcal{T}^\lambda$. Consequently $\mathrm{A}(\mathcal{T}) = \bigcup_{\lambda \in \mathfrak{R}} \mathcal{A}_\lambda(\mathcal{T})$ is quasi–locally presentable, and weakly κ-generated. $\square$

Theorem 3.3. *If $\mathcal{T}$ is a well generated triangulated category, then every functor $F : \mathrm{A}(\mathcal{T}) \to \mathcal{A}b$ which is contravariant, exact and sends coproducts into products is representable.*

Proof. Without losing the generality we may assume that $\mathcal{T}$ is well κ–generated, for some $\kappa \geq \aleph_1$ (if not, we replace κ by $\aleph_1$). By Proposition 3.2, $\mathrm{A}(\mathcal{T})$ is a weakly κ–generated quasi–locally presentable category. In order to apply Theorem 2.3 we have only to show that every λ–presentable object X of $\mathrm{A}_\lambda(\mathcal{T})$ admits an embedding into an object in $S \in \mathrm{A}_\lambda(\mathcal{T})$ which is λ–presentable in $\mathrm{A}_\lambda(\mathcal{T})$ and injective in $\mathrm{A}(\mathcal{T})$. But this follows immediately from Lemma 3.1, since, according to [11, Corollary 5.1.23], every $X \in \mathrm{mod}(\mathcal{T}^\lambda)$ admits an embedding into an object of the form $H(x)$ with $x \in \mathcal{T}^\lambda$. $\square$

Note that the category $\mathrm{A}(\mathcal{T})$ is usually "huge", in the sense that it is not well (co)powered, as we learned on [11, Appendix C]. Thus Proposition 3.2

and Theorem 3.3 provide an example of such a huge category which is quasi–locally presentable and for which representability Theorem 2.3 applies.

Combining Theorem 3.3 with Corollary 1.2 we obtain a new proof of Brown representability theorem for triangulated well generated categories:

Corollary 3.4. *Every cohomological functor, which sends coproducts into products, defined on a well generated triangulated category with values in abelian groups is representable.*

Example 3.5. Recall from [14] the definition: A triangulated category with coproducts is called *locally well generated*, provided that every localizing subcategory which is generated (in the triangulated sense) by a set of objects is well generated. The typical example of a locally well generated triangulated category, which is not well generated, is the homotopy category $\mathbf{K}(\mathrm{Mod}R)$ where R is a ring which is not pure–semisimple (see [14, Theorem 3.5]). Objects in this category are complexes of R-modules, and maps are classes of homotopy equivalent maps of complexes.

Let consider $R = \mathbb{Z}$, so $\mathcal{T} = \mathbf{K}(\mathcal{A}b)$ is locally well generated, but not well generated. Then there is a non–representable exact contravariant functor $F : \mathrm{A}(\mathbf{K}(\mathcal{A}b)) \to \mathcal{A}b$, which sends coproducts into products. For showing this, let denote $\mathcal{A}b_\kappa$ the closure under direct sums and direct summands of the full subcategory of $\mathcal{A}b$ consisting of all groups of cardinality smaller that κ, for every $\kappa \in \mathfrak{R}$. Clearly, every $A \in \mathcal{A}b$ has an $\mathcal{A}b_\kappa$-precover, what means a group homomorphism $X_A \to A$ with $X_A \in \mathcal{A}b_\kappa$ such that the induced map $\mathcal{A}b(X, X_A) \to \mathcal{A}b(X, A)$ is surjective for all $X \in \mathcal{A}b_\kappa$. For every $A \in \mathcal{A}b$, we construct the complex

$$Y_{A,\kappa} : \ \cdots \to Y_\kappa^{-2} \to Y_\kappa^{-1} \to Y_\kappa^0 \to A \to 0,$$

whose differentials are the composition $Y_\kappa^{-n} \to A^{-n} \to Y_\kappa^{-n+1}$, where A^{-n} is the kernel of $Y_\kappa^{-n+1} \to A^{-n+1}$ (and $A^0 = A$), and $Y_\kappa^{-n} \to A^{-n}$ is an $\mathcal{A}b_\kappa$-precover of A^{-n}, for all $n \geq 0$. As it may be seen in [10, Example 11], the functor:

$$F = \prod_{\kappa \in \mathfrak{R}} \mathcal{T}(-, Y_{\mathbb{Z}^\kappa, \kappa}) : \mathbf{K}(\mathcal{A}b) \to \mathcal{A}b$$

is cohomological, sends coproducts into products but is not representable. Note that the argument showing that this functor is well defined is similar to the one used in Example 2.5. By Lemma 1.1 the functor

$$F^* : \mathrm{A}(\mathbf{K}(\mathcal{A}b)) \to \mathcal{A}b, F^*(X) = \mathrm{Hom}_\mathcal{T}\left(X, \prod_{\kappa \in \mathfrak{R}} \mathcal{T}(-, Y_{\mathbb{Z}^\kappa, \kappa})\right)$$

is contravariant, exact, sends coproducts into products, but is not representable.

References

[1] J. Adámek and J. Rosický *Locally presentable and accessible categories*, Cambridge University Press, 1994.

[2] S. Breaz, G.C. Modoi, *A reformulation of Brown representability theorem*, Mathematica(Cluj), **51**(2009), 129–133.

[3] J. Franke, *On the Brown representability theorem for triangulated categories*, Topology, **40**(2001), 667–680.

[4] P. Gabriel, F. Ulmer, *Lokal präsentierbare Kategorien*, Springer Lecture Notes in Math., **221**, Berlin–Heidelberg, 1971.

[5] H. Krause, *Smashing subcategories and the telescope conjecture – an algebraic approach*, Invent. Math. **139**(2000), 99–133.

[6] H. Krause, *On Neeman's well generated triangulated categories*, Documenta Math. **6**(2001), 121–126.

[7] H. Krause, *Localization theory for triangulated categories*, to appear in the proceedings of the *Workshop on Triangulated Categories*, Leeds 2006.

[8] S. Maclane, *Categories for the Working Mathematicians*, Springer Verlag, New York, Heidelberg, Berlin, 1971.

[9] G. C. Modoi, *On perfectly generating projective classes in triangulated categories*, Comm. Algeba, 38(2010), 995–1011.

[10] G.C. Modoi, J. Šťovíček, *Brown representability often fails for homotopy categories of complexes*, preprint, arXiv:1012.4109v1 [math.CT].

[11] A. Neeman, *Triangulated Categories*, Annals of Mathematics Studies, **148**, Princeton University Press, Princeton, NJ, 2001.

[12] A. Neeman, *Brown Representability follows by Rosický's theorem*, J. Topology, **2**(2009), 262–276.

[13] N. Popescu, L. Popescu, *Theory of Categories*, Editura Academiei, Bucureşti and Sijthoff & Noordhoff International Publishers, 1979.

[14] J. Šťovíček, *Locally well generated homotopy categories of complexes*, preprint, arXiv: 0810.5684v1.

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